Existence of quasi-invariance flow on loop space over a general non-compact manifold

Xin Chen

Shanghai Jiao Tong University

The 16th Workshop on Markov Processes and Related Topics
July 12–16, 2021, Central China University

Based on a joint work with Xue-Mei Li and Bo Wu

Outline

1. **Introduction**
   - Riemannian path space
   - Riemannian loop space
   - Functional inequalities on $L_o(M)$
2. **Main Results**

Outline

1. **Introduction**
   - Riemannian path space
   - Riemannian loop space
   - Functional inequalities on $L_o(M)$
2. **Main Results**

Riemannian path space

- M: a complete Riemannian manifold;
- $P_o(M) := \{\gamma \in C([0,1]; M); \gamma(0) = o\}$;
- μ_o: the Brownian measure on $P_o(M)$ (the probability measure on $P_o(M)$ under which the distribution of $\gamma(\cdot)$ is an M-valued Brownian motion;
- $U_\cdot(\gamma)$: stochastic horizontal lift along $\gamma(\cdot)$;

Riemannian path space

- Cameron-Martin space:
  $\mathbb{H} := \{h \in C([0,1];\mathbb{R}^d); h(0) = 0, \int_0^1 |\dot{h}(s)|^2 ds < \infty\};$

- Malliavian derivative

$$F(\gamma) = f\big(\gamma(t_1), \cdots \gamma(t_k)\big),\ 0 < t_1 < \cdots < t_k < 1,\ f \in C_b^1(M^k);$$

$$D_h F(\gamma) = \sum_{i=1}^{k} \Big\langle \nabla_i f\big(\gamma(t_1), \cdots \gamma(t_k)\big), U_{t_i}(\gamma) h(t_i) \Big\rangle_{\gamma(t_i)}, \quad h \in \mathbb{H};$$

$DF(\gamma) \in \mathbb{H}$ such that

$$\langle DF(\gamma), h\rangle_{\mathbb{H}} = D_h F(\gamma), \quad h \in \mathbb{H};$$

Riemannian path space

- O-U Dirichlet form (on $L^2(P_o(M);\mu)$)

$$\mathscr{E}(F,F) = \int_{P_o(M)} \|DF(\gamma)\|_{\mathbb{H}}^2 \mu_o(d\gamma), \quad F \in \mathscr{D}(\mathscr{E});$$

- Motivation: Feynman path integral;
- Motivation: SPDE($P(M)$-valued process), Quasi-regularity([Z.M. Ma and M. Röckner 92])

Existence of quasi-invariance flow

For every $h \in \mathbb{H}$, there exists a flow $\xi^\varepsilon : P_o(M) \to P_o(M)$, $\varepsilon \in \mathbb{R}$ such that the following statements hold.

- $\xi^0(\gamma) = \gamma$;
- There exists a μ_o-null set Λ_0, such that for all $\gamma \in P_o(M)/\Lambda_0$, $\xi_s^\varepsilon(\gamma)$ is jointly C^1 in $\varepsilon \in \mathbb{R}$ and continuous in $s \in [0,1]$ and

$$\frac{\partial \xi_s^\varepsilon(\gamma)}{\partial \varepsilon} = U_s(\xi_\cdot^\varepsilon)h(s),\ \varepsilon \in \mathbb{R},\ s \in [0,1],$$

- For every $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, it holds that for μ_o-a.s. $\gamma \in P_o(M)$,

$$\xi^{\varepsilon_1} \circ \xi^{\varepsilon_2}(\gamma) = \xi^{\varepsilon_1+\varepsilon_2}(\gamma);$$

- For every $\varepsilon \in \mathbb{R}$, suppose that μ_o^ε is the law of $\xi_\cdot^\varepsilon$ on $P_o(M)$. Then μ_o^ε is absolutely continuous with respect to μ_o.

Existence of quasi-invariance flow

- M compact, $h \in C^1([0,1];\mathbb{R}^d)$: [B. Driver 92]
- M compact, $h \in \mathbb{H}$: [E. Hsu 95]
- M complete and stochastically complete, $h \in \mathbb{H}$: [E. Hsu and C. Ouyang 09]

Motivation: To define the gradient for a general $F \in H^1$.

Outline

1. **Introduction**
   - Riemannian path space
   - Riemannian loop space
   - Functional inequalities on $L_o(M)$
2. **Main Results**

- $L_o(M) := \{\gamma \in C([0,1];M); \gamma(0) = \gamma(1) = o\}$;
- ν_o: the Brownian bridge measure on $P_o(M)$

$$\mathbb{E}_{\nu_o}[F] = \mathbb{E}_{\mu_o}[F|\gamma(1) = o];$$

- Finite dimensional expression

$$F(\gamma) = f\big(\gamma(t_1), \cdots \gamma(t_k)\big),\ 0 < t_1 < \cdots < t_k < 1,\ f \in C_b^1(M^k);$$

$$\mathbb{E}_{\nu_o}[F] = \frac{\int_{M^k} f(x_1, \cdots, x_k) p(t_1, o, x_1) \cdots p(1 - t_k, x_k, o) dx_1 \cdots dx_k}{p(1, o, o)}; \tag{1}$$

Riemannian loop space

- $U_\cdot(\gamma)$: stochastic horizontal lift along $\gamma(\cdot)$

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dB_t^i + H_{\nabla \log p(1-t,\gamma(t),o)}(U_t)dt;$$

- $\mathbb{H}_0 := \{h \in C([0,1];\mathbb{R}^d); h(0) = h(1) = 0, \int_0^1 |\dot{h}(s)|^2 ds < \infty\};$

- Malliavian derivative

$$F(\gamma) = f\big(\gamma(t_1), \cdots \gamma(t_k)\big),\ 0 < t_1 < \cdots < t_k < 1,\ f \in C_b^1(M^k);$$

$$D_{0,h}F(\gamma) = \sum_{i=1}^{k} \Big\langle \nabla_i f\big(\gamma(t_1), \cdots \gamma(t_k)\big), U_{t_i}(\gamma)h(t_i) \Big\rangle_{\gamma(t_i)},\quad h \in \mathbb{H}_0;$$

- Malliavin gradient operator: $DF(\gamma) \in \mathbb{H}_0$ such that

$$\langle D_0 F(\gamma), h\rangle_{\mathbb{H}_0} = D_{0,h}F(\gamma), \ \ h \in \mathbb{H}_0;$$

- O-U Dirichlet form (on $L^2(L_o(M);\nu)$)

$$\mathscr{E}_0(F,F) = \int_{L_o(M)} \|D_0 F(\gamma)\|^2_{\mathbb{H}_0} \nu_o(d\gamma), \ \ F \in \mathscr{D}(\mathscr{E}_0);$$

- Motivation: homology and cohomology, string theory and loop quantum gravity;

Outline

1. **Introduction**
   - Riemannian path space
   - Riemannian loop space
   - Functional inequalities on $L_o(M)$

2. **Main Results**

Functional inequalities on $L_o(M)$

- For functional inequalities for $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ (on $L^2(P_o(M); \mu_o)$), the crucial ingredients is the Ricci curvature bound on M;
- [L. Gross 91], [S. Aida 98] If M is not simply connected, then weak Poincaré inequality does not hold for $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$.
- [A. Eberle 03] There exists a simply connected compact manifold M, such that Poincaré inequality does not hold for $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ on $L_o(M)$.
- [S. Aida 98] If M is compact and simply connected, then weak Poincaré inequality holds for $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$, but any estimation of the rate function is unknown.

Functional inequalities on $L_o(M)$

- [L. Gross 91], [F.Z. Gong, Z.M. Ma 98] If M is compact and simply connected, then there exists a $V \in \bigcap_{p=1}^{\infty} L^p(L_o(M);\nu_o)$

$$\mathbb{E}_{\nu_o}[F^2 \log F^2] - \mathbb{E}_{\nu_o}[F^2] \log \mathbb{E}_{\nu_o}[F^2] \leqslant C\mathscr{E}_0(F,F) + \mathbb{E}_{\nu_o}[VF^2]$$

- [F.Z. Gong, M. Röckner, L.M. Wu 01], [S. Aida 01] If M is compact simply connected, then there exists a ground state measure $\nu_{o,\phi}$ such that

$$\mathbb{E}_{\nu_{o,\phi}}[F^2] - \mathbb{E}_{\nu_{o,\phi}}[F]^2 \leqslant C \int_{L_o(M)} \|D_0 F(\gamma)\|_{\mathbb{H}_0}^2 \nu_{o,\phi}(d\gamma);$$

- [S. Aida 99] If $M = H^n$, then

$$\mathbb{E}_{\nu_o}[F^2 \log F^2] - \mathbb{E}_{\nu_o}[F^2] \log \mathbb{E}_{\nu_o}[F^2] \leqslant C \int_{L_o(H^n)} \rho(\gamma)^2 \|D_0 F(\gamma)\|_{\mathbb{H}_0}^2 \nu_o(d\gamma),$$

where $\rho(\gamma) = \sup_{t\in[0,1]} d(o, \gamma(t))$.

- [C., X.M. Li, B. Wu 10], [S. Aida 17] If $M = H^n$, then

$$\mathbb{E}_{\nu_o}[F^2] - \mathbb{E}_{\nu_o}[F]^2 \leqslant C\mathscr{E}_0(F, F);$$

- [C., X.M. Li, B. Wu 11] If M is compact and $Ric > 0$, then for every $\delta > 0$

$$\mathbb{E}_{\nu_o}[F^2] - \mathbb{E}_{\nu_o}[F]^2 \leqslant Cr^{-\delta}\mathscr{E}_0(F,F) + r\|F\|_\infty^2.$$

- [A. Eberle 03, S. Aida 11] If M is compact, then there exists a $r_0 > 0$, such that for every $F \in \mathscr{D}(\mathscr{E}_0)$ with $\mathrm{supp}F \subseteq \mathbb{B}_o(r_0)$,

$$\mathbb{E}_{\nu_o}[F^2 \log F^2] - \mathbb{E}_{\nu_o}[F^2] \log \mathbb{E}_{\nu_o}[F^2] \leqslant C \int_{L_o(M)} \|D_0 F(\gamma)\|_{\mathbb{H}_0}^2 \nu_o(d\gamma).$$

Here $\mathbb{B}_o(r_0) := \{\gamma \in L_o(M); \sup_{t\in[0,1]} d(o,\gamma(t)) < r_0\}$;

Important ingredients

- Topological properties of based manifold M;
- M compact, asymptotic gradients estimates for heat kernel

$$|\nabla_x \log p(t,x,y)| \leqslant C\left(\frac{d(x,y)}{t} + \frac{1}{\sqrt{t}}\right),\ x,y \in M,\ t \in (0,1].$$

$$\left|\nabla_x^2 \log p(t,x,y)\right| \leqslant C\left(\frac{d^2(x,y)}{t^2} + \frac{1}{t}\right),\ x,y \in M,\ t \in (0,1].$$

- [P. Malliavin, D.W. Stroock, 97]
  Suppose $y \in M$ and $K \subset \mathrm{Cut}^c(y)$ is a compact set, then

$$\limsup_{t\downarrow 0} \sup_{x\in K} \left| t\nabla_x^2 \log p(t,x,y) + \nabla_x^2\left(\frac{d^2(x,y)}{2}\right)\right| = 0.$$

[B.Driver 94] Suppose M is compact, then for every $h \in \mathbb{H}_0 \cap C^1$, there exists a flow $\xi^\varepsilon : L_o(M) \to L_o(M)$, $\varepsilon \in \mathbb{R}$ such that the following statements hold.

- $\xi^0(\gamma) = \gamma$. There exists a ν_o-null set Λ_0, such that for all $\gamma \in L_o(M)/\Lambda_0$, $\xi_s^\varepsilon(\gamma)$ is jointly C^1 in $\varepsilon \in \mathbb{R}$ and continuous in $s \in [0,1]$ and

$$\frac{\partial \xi_s^\varepsilon(\gamma)}{\partial \varepsilon} = U_s(\xi_\cdot^\varepsilon)h(s),\ \varepsilon \in \mathbb{R},\ s \in [0,1];$$

- For every $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, it holds that for ν_o-a.s. $\gamma \in L_o(M)$,

$$\xi^{\varepsilon_1} \circ \xi^{\varepsilon_2}(\gamma) = \xi^{\varepsilon_1+\varepsilon_2}(\gamma);$$

- For every $\varepsilon \in \mathbb{R}$, suppose that ν_o^ε is the law of $\xi_\cdot^\varepsilon$ on $L_o(M)$. Then ν_o^ε is absolutely continuous with respect to ν_o

Some Questions

- What is the functional inequalities for $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ on $L_o(M)$ when M is non-compact M?
- Whether there exists a quasi-invarince flow on $L_o(M)$ when M is non-compact and $h \in \mathbb{H}_0$.

Asymptotic gradient estimates for heat kernel

Theorem (C., Li and Wu 21+)

Suppose M is a general complete Riemannian manifold.

(1) *Suppose $x, y \in M$ and $x \notin Cut(y)$, then*

$$\lim_{t \downarrow 0} t \nabla_x \log p(t,x,y) = -\nabla_x \left(\frac{d^2(x,y)}{2} \right).$$

Here the convergence is uniformly for $x \in K$ with K being a compact subset of $Cut^c(y)$.

(2) *Suppose $K \subset M$ is a compact subset of M, then there exists a positive constant $C(K)$, (which depends on K) such that*

$$|\nabla_x \log p(t,x,y)| \leqslant C \left(\frac{d(x,y)}{t} + \frac{1}{\sqrt{t}} \right), \quad x, y \in K, \ t \in (0,1].$$

Asymptotic gradient estimates for heat kernel

Theorem (C., Li and Wu 21+)

Suppose M is a general complete Riemannian manifold.

(1) *Suppose $y \in M$ and $K \subset Cut^c(y)$ is a compact set, then*

$$\limsup_{t \downarrow 0} \sup_{x \in K} \left| t \nabla_x^2 \log p(t,x,y) + \nabla_x^2 \left(\frac{d^2(x,y)}{2} \right) \right| = 0.$$

(2) *Suppose $K \subset M$ is a compact subset of M, then there exists a positive constant $C(K)$, such that for all $x, y \in K$, $t \in (0,1]$,*

$$\left| \nabla_x^2 \log p(t,x,y) \right| \leqslant C \left(\frac{d^2(x,y)}{t^2} + \frac{1}{t} \right).$$

X. Chen, X.-M. Li, B. Wu: Logarithmic heat kernels: estimates without curvature restrictions, arXiv:2106.02746.

Theorem (C., Li and Wu 21+)

Suppose M is complete and stochastically complete, there exists a (Brownian bridge) probability measure ν_o on $L_o(M)$ which posses the finite dimensional distribution (1).
Moreover, given a $\alpha \in (0, \frac{1}{2})$, there exists a ν_o-null set $\Delta \in L_o(M)$, such that for every $\gamma \notin \Delta$,

$$d(\gamma(s), \gamma(t)) \leqslant C(\gamma)(t-s)^{\alpha}, \ 0 \leqslant s \leqslant t \leqslant 1.$$

Theorem (C., Li and Wu 21+)

Suppose M is complete and stochastically complete. The quadratic form $(\mathscr{E}_0, \mathscr{F}C_b)$ is closable, and its closed extension $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$ is a quasi-regular Dirichlet form.

Functional inequalities

Theorem (C., Li and Wu 21+)

If M is complete and stochastically complete, then there exists a $r_0 > 0$, such that for every $F \in \mathscr{D}(\mathscr{E}_0)$ with $\mathrm{supp} F \subseteq \mathbb{B}_o(r_0)$,

$$\mathbb{E}_{\nu_o}[F^2 \log F^2] - \mathbb{E}_{\nu_o}[F^2] \log \mathbb{E}_{\nu_o}[F^2] \leqslant C \int_{L_o(M)} \|D_0 F(\gamma)\|^2_{\mathbb{H}_0} \nu_o(d\gamma).$$

Here $\mathbb{B}_o(r_0) := \{\gamma \in L_o(M); \sup_{t\in[0,1]} d(o, \gamma(t)) < r_0\}$;

Theorem (C., Li and Wu 21+)

Suppose M is complete and stochastically complete, then there exists a positive function $V \in \cap_{p=1}^{\infty} L^p(\mathbb{B}_o(R); \nu_o)$ for all $R > 0$, such that

$$Ent_{\nu_o}(F^2) \leqslant C\mathscr{E}_0(F, F) + \mathbb{E}_{\nu_o}(VF^2), \quad F \in \mathscr{D}_{loc}(\mathscr{E}_0),$$

where $\mathscr{D}_{loc}(\mathscr{E}_0) := \{F \in \mathscr{D}(\mathscr{E}_0); \mathrm{supp}(F) \subseteq \mathbb{B}_o(R)$ for some $R > 0\}$.

Theorem (C., Li and Wu 21+)

Suppose M is complete and stochastically complete, then for every $h \in \mathbb{H}_0$, there exists a flow $\xi^\varepsilon : L_o(M) \to L_o(M)$, $\varepsilon \in \mathbb{R}$ such that

- $\xi^0(\gamma) = \gamma$ *and*

$$\frac{\partial \xi_s^\varepsilon(\gamma)}{\partial \varepsilon} = U_s(\xi_\cdot^\varepsilon)h(s),\ \varepsilon \in \mathbb{R},\ s \in [0,1];$$

- *For every $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, it holds that for ν_o-a.s. $\gamma \in L_o(M)$,*

$$\xi^{\varepsilon_1} \circ \xi^{\varepsilon_2}(\gamma) = \xi^{\varepsilon_1+\varepsilon_2}(\gamma);$$

- *Suppose that ν_o^ε is the law of $\xi_\cdot^\varepsilon$ on $L_o(M)$. Then*

$$\frac{d\nu_o^\varepsilon}{d\nu_o} = \exp\left[\int_0^\varepsilon \Phi_h^1(\xi^{-\lambda})d\lambda\right],$$

where

$$\Phi_h^t(\gamma) := \int_0^t \left\langle h'(s) + \frac{1}{2}Ric_{U_s(\gamma)}h(s), d\beta_s(\gamma)\right\rangle$$

Idea of the proof: cut-off vector fields

[A. Thalmaier 97], [A. Thalmaier, F.Y. Wang 98] Let $D_m \subseteq M$ is a increasing sequence of subset with $\cup_m D_m = M$, set $\tau_m := \tau_{D_m}$. For any $m \in \mathbb{N}$, there exists a random vector field $l_m : [0,1] \times C([0,1]; M) \to [0,1]$, such that

- $l_m(t,\gamma) = \begin{cases} 1, & t \leqslant \tau_{m-1}(\gamma) \wedge 1 \\ 0, & t > \tau_m(\gamma) \end{cases}$.;
- $l_m(t,\gamma)$ is $\mathscr{F}_t^\gamma$-adapted and $l_m(\cdot,\gamma)$ is absolutely continuous;
- For every positive integer $k \in \mathbb{Z}_+$, we have

$$\sup_{x \in D_m} \mathbb{E}_{\mu_x}\Big[\int_0^1 |l_m'(s,\gamma)|^k ds\Big] \leqslant C(m,k).$$

Idea of the proof

- For any $t < 1$,

$$\frac{d\nu_o^\varepsilon}{d\nu_o}\Big|_{\mathscr{F}_t} = \frac{p\left(1-t, \xi_t^{-\varepsilon}, o\right)}{p(1-t, \gamma(t), o)} \exp\left[\int_0^\varepsilon \Phi_h^t\left(\xi^{-\lambda}\right) d\lambda\right]$$
$$=:= Z_{h,1}^{\varepsilon,t}(\gamma) Z_{h,2}^{\varepsilon,t}(\gamma)$$

-
$$\beta_s(\xi^\lambda) = \int_0^s U_r^{-1}(\xi^\lambda) \circ d\xi_r^\lambda = \int_0^s O_r^{h,\lambda} d\beta_r(\gamma) + \int_0^s A_r^{h,\lambda} dr,$$

where $O_\cdot^{h,\lambda}$ is $SO(n)$-valued process and

$$\sup_{\gamma \in \mathbb{B}_o(r)} \left|A_s^{h,\varepsilon}(\gamma)\right| \leqslant c_1(\varepsilon, r)\left(|h'(s)| + |h(s)|\right) < \infty,\ \forall\, \varepsilon \in \mathbb{R},\ r > 0.$$

-
$$\beta_s(\gamma) = b_s + \int_0^s U_r^{-1}(\gamma) \nabla \log p\left(1-r, \gamma(r), o\right) dr.$$

Idea of the proof

- If M is compact, then we have

$$\left|\Phi_h^t\left(\xi^{-\lambda}\right)\right| \leqslant c_1 \int_0^t \frac{d(o,\gamma(s))}{1-s} ds,$$

and there exists a $\lambda > 0$,

$$\nu_{o,o}\left[\exp\left(\lambda\left(\int_0^1 \frac{d(o,\gamma(s))}{1-s} ds\right)^2\right)\right] < \infty.$$

- When M is non-compact, the above estimates do not hold.
- Instead, we are going to prove (convergence in probability)

$$\lim_{t\uparrow 1} Z_{h,1}^{\varepsilon,t} = 1, \ \lim_{t\uparrow 1} Z_{h,2}^{\varepsilon,t} = Z_{h,2}^{\varepsilon,1}.$$

Idea of the proof

- For $\mathscr{F}_t$ adapted F we have

$$\int_{L_o(M)} F d\nu_o^\varepsilon = \int_{L_o(M)} F Z_{h,1}^{\varepsilon,t} Z_{h,2}^{\varepsilon,t} d\nu_o.$$

- Taking $t \uparrow 1$, by Fatou's lemma we know for every non-negative F

$$\int_{L_o(M)} F d\nu_o^\varepsilon \geqslant \int_{L_o(M)} F Z_{h,2}^{\varepsilon,1} d\nu_o.$$

Idea of the proof

- Now we take $G(\gamma) = F\left(\xi^{\varepsilon}(\gamma)\right) Z_{h,2}^{\varepsilon,1}\left(\xi^{\varepsilon}(\gamma)\right)$ obtain

$$
\begin{aligned}
&\int_{L_o(M)} F(\gamma)\, Z_{h,2}^{\varepsilon,1}(\gamma)\, \nu_o(d\gamma) \\
&= \int_{L_o(M)} G\left(\xi^{-\varepsilon}(\gamma)\right) \nu_o(d\gamma) \\
&= \int_{L_o(M)} G(\gamma)\, \nu_o^{-\varepsilon}(d\gamma) \geqslant \int_{L_o(M)} G(\gamma) Z_{h,2}^{-\varepsilon,1}(\gamma) \nu_o(d\gamma) \\
&= \int_{L_o(M)} F\left(\xi^{\varepsilon}(\gamma)\right) Z_{h,2}^{\varepsilon,1}\left(\xi^{\varepsilon}(\gamma)\right) Z_{h,2}^{-\varepsilon,1}(\gamma) \nu_o(d\gamma) \\
&= \int_{L_o(M)} F\left(\xi^{\varepsilon}(\gamma)\right) \nu_o(d\gamma) = \int_{L_o(M)} F(\gamma) \nu_o^{\varepsilon}(d\gamma).
\end{aligned}
$$

Idea of the proof

- $$\begin{aligned} & Z_{h,2}^{\varepsilon,1}\left(\xi^{\varepsilon}(\gamma)\right) Z_{h,2}^{-\varepsilon,1}(\gamma) \\ & = \exp\left[\int_0^{\varepsilon} \Phi_h\left(\xi^{-\lambda}\circ\xi^{\varepsilon}(\gamma)\right) d\lambda + \int_0^{-\varepsilon} \Phi_h\left(\xi^{-\lambda}(\gamma)\right) d\lambda\right] \\ & = \exp\left[\int_0^{\varepsilon} \Phi_h\left(\xi^{-\lambda+\varepsilon}(\gamma)\right) d\lambda - \int_0^{\varepsilon} \Phi_h\left(\xi^{\lambda}(\gamma)\right) d\lambda\right] = 1. \end{aligned}$$

Idea of the proof

The limit for $Z_{h,1}^{\varepsilon,t}$

- Comparison with the heat kernel on compact manifold

$$\frac{p\left(1-t,\xi_t^{-\varepsilon}(\gamma),o\right)}{p\left(1-t,\gamma(t),o\right)}1_{\{\tau_m(\gamma)>t\}} \leqslant \frac{p_{\tilde{M}_{m_1}}\left(1-t,\xi_t^{-\varepsilon}(\gamma),o\right)+e^{-\frac{L}{1-t}}}{p_{\tilde{M}_{m_1}}\left(1-t,\gamma(t),o\right)-e^{-\frac{L}{1-t}}}1_{\{\tau_m(\gamma)>t\}}$$

$$\frac{p\left(1-t,\xi_t^{-\varepsilon}(\gamma),o\right)}{p\left(1-t,\gamma(t),o\right)}1_{\{\tau_m(\gamma)>t\}} \geqslant \frac{p_{\tilde{M}_{m_1}}\left(1-t,\xi_t^{-\varepsilon}(\gamma),o\right)-e^{-\frac{L}{1-t}}}{p_{\tilde{M}_{m_1}}\left(1-t,\gamma(t),o\right)+e^{-\frac{L}{1-t}}}1_{\{\tau_m(\gamma)>t\}}$$

Idea of the proof

The limit for $Z^{\varepsilon,t}_{h,2}$

- $$\begin{aligned}\Phi^t_h(\xi^\lambda) = &\int_0^t \left\langle \left(h'(s) + \frac{1}{2}\mathrm{Ric}_{U_s(\xi^\lambda)}h(s)\right), O^{h,\lambda}_s db_s + A^{h,\lambda}_s ds \right\rangle \\ &+ \int_0^t \left\langle h'(s), O^{h,\lambda}_s U^{-1}_s(\gamma)\nabla \log p\left(1-s,\gamma(s),o\right) ds \right\rangle \\ &+ \int_0^t \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\xi^\lambda)}h(s), O^{h,\lambda}_s U^{-1}_s(\gamma)\nabla \log p\left(1-s,\gamma(s),o\right) ds \right\rangle \\ &:= I^{1,\lambda}_t + I^{2,\lambda}_t + I^{3,\lambda}_t.\end{aligned}$$

- The difficult one: the limit for $I^{2,\lambda}_t$.

Idea of the proof

The limit for $Z_{h,2}^{\varepsilon,t}$

- $$\begin{aligned}\Phi_h^t(\xi^\lambda) = &\int_0^t \left\langle \left(h'(s) + \frac{1}{2}\mathrm{Ric}_{U_s(\xi^\lambda)}h(s)\right), O_s^{h,\lambda}db_s + A_s^{h,\lambda}ds \right\rangle \\ &+ \int_0^t \left\langle h'(s), O_s^{h,\lambda}U_s^{-1}(\gamma)\nabla \log p\left(1-s,\gamma(s),o\right)ds \right\rangle \\ &+ \int_0^t \left\langle \frac{1}{2}\mathrm{Ric}_{U_s(\xi^\lambda)}h(s), O_s^{h,\lambda}U_s^{-1}(\gamma)\nabla \log p\left(1-s,\gamma(s),o\right)ds \right\rangle \\ &:= I_t^{1,\lambda} + I_t^{2,\lambda} + I_t^{3,\lambda}.\end{aligned}$$

- The difficult one: the limit for $I_t^{2,\lambda}$.

Idea of the proof

$$d\left(O_s^{h,\lambda}U_s^{-1}\nabla\log p\left(1-s,\gamma(s),o\right)\right)=L_s^{1,h,\lambda}db_s+L_s^{2,h,\lambda}ds,$$

where

$$\sup_{\gamma\in\mathbb{B}_o(r)}\left|L_s^{1,h,\lambda}(\gamma)\right|\leqslant c_3(r)\left(\frac{d\left(\gamma(s),o\right)^2}{(1-s)^2}+\frac{1}{1-s}+\frac{|h(s)|d\left(\gamma(s),o\right)}{1-s}\right),$$

$$\sup_{\gamma\in\mathbb{B}_o(r)}\left|L_s^{2,h,\lambda}(\gamma)\right|\leqslant c_3(r)\left(\frac{|h(s)|d\left(\gamma(s),o\right)^2}{(1-s)^2}\right.$$

$$\left.+\frac{|h(s)|}{1-s}+\left(1+|h'(s)||h(s)|\right)\cdot\left(\frac{d\left(\gamma(s),o\right)}{1-s}+\frac{1}{\sqrt{1-s}}\right)\right).$$

$$
\begin{aligned}
I_t^{2,\lambda} &= \left\langle h(t), O_t^{h,\lambda} U_t^{-1}(\gamma)\nabla \log p\left(1-t,\gamma(t),o\right)\right\rangle \\
&\quad - \int_0^t \left\langle h(s), d\left(O_s^{h,\lambda} U_s^{-1}\nabla \log p\left(1-s,\gamma(s),o\right)\right)\right\rangle \\
&\quad = \left\langle h(t), O_t^{h,\lambda} U_t^{-1}(\gamma)\nabla \log p\left(1-t,\gamma(t),o\right)\right\rangle \\
&\quad - \int_0^t \langle h(s), L_s^{1,h,\lambda} db_s\rangle - \int_0^t \langle h(s), L_s^{2,h,\lambda} ds\rangle \\
&\quad := I_t^{21,\lambda} + I_t^{22,\lambda} + I_t^{23,\lambda}.
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\uparrow 1} \nu_o\left[|I_t^{21,\lambda}| 1_{\{\tau_m>t\}}\right] \\
&\leqslant c_4 \lim_{t\uparrow 1} |h(t)|\nu_o\left[\left(\frac{d\left(\gamma(t),o\right)}{1-t}+\frac{1}{\sqrt{1-t}}\right)1_{\{\gamma(t)\in B_o(m)\}}\right] \\
&\leqslant \lim_{t\uparrow 1}\frac{c_5|h(t)|}{\sqrt{1-t}} \leqslant c_5 \lim_{t\uparrow 1}\left(\int_t^1 |h'(s)|^2 ds\right)^{1/2} = 0.
\end{aligned}
$$

Thank you for your attention!